LECTURE 5 - MIXING PROPERTIES OF SFTS

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For $d = 1$, there are only a few mixing properties of SFTs that one might care about: Topological mixing and topological transitivity.

However for $d > 1$ (or for non-SFT shifts in $d = 1$) there are host of mixing properties; each coming with a different bent and application in mind. We will focus on a few chosen ones and talk about the applications.

Sorry for the very long list. It is not to be taken as a list of instruction but some opinion on what might be. You have the freedom to mix and match according to your liking. Just try to give me some idea before hand.

(1) Define Strong irreducibility (SI) of shifts space and block-gluing (definitions can be found in [7]). If you have time, you should also define TSSM [1]. State some examples (which appear later as well).
(2) Talk about why they are interesting:
  (a) If a subshift is SI then X is a entropy minimal, meaning that any proper subshift has strictly smaller entropy [13, Lemma 2.7]. (He proved this under a weaker condition but say that you don't want to get into it). (I will probably say this in some form in an earlier talk so just keep this to a line).
  (b) If an SFT X is SI then a version of Krieger's embedding theorem was proved for it by Samuel Lightwood [10, 11], that is, in such subshifts we can embed any other subshift without periodic points and strictly smaller entropy.
  (c) Following work by Şahin [12], Chandgotia and Meyerovitch prove in [8] that block-gluing SFTs are universal in the sense that they can model every Borel dynamical system of smaller entropy. (I will discuss these later so maybe you can skip this or just say, not write)
  (d) Block gluing subshifts factor onto all full shifts (and more) of smaller entropy [2].
  (e) Mixing properties have a host of implications on the computability of thermodynamic quantities like entropy and pressure. From instance the entropy of an SI shift is computable [9]. Better mixing properties like TSSM help to approximate these quantities more efficiently [4, 3].

  You do not have to say everything. People should already know what is entropy, embedding, factors, measures of maximal entropy by now so just keep a line of definition and say that it is a recall. It should leave an impression of why it is important but not necessarily overwhelm people with results.

(3) Now you can start giving examples. I will define hom-shifts earlier. Recall what they are hom-shifts. These are the space of graph homomorphisms from $\mathbb{Z}^d$ to a fixed undirected graph H (maps from vertices to vertices which preserve adjacency). Say a little bit about why hom-shifts are always topologically transitive when the graph H is connected [5, Slides 18-30].

(4) State the general question.

  (a) Begin with the following question:

  Question 1. *Given a graph H, when is the space of graph homomorphisms from $\mathbb{Z}^d$ to H SI?*

  (b) Explain why the hard-core shift is SI. Explain why 3-colourings are not block-gluing. You can look at notes in [6]. You do not have to introduce all the jargon frankly.

  (c) Say that the argument for 3-colourings generalises to the following setting: Given a connected graph H, a surjective graph homomorphism from $f : H' \to H$ is a *cover* if for all $v \in H'$, $f|_{\{w\in H' \,:\, w\sim_{H'} v\}}$ is bijective onto $\{w \in H \;:\; w \sim_H f(v)\}$. Say that among the covers, there is a universal cover $f_{uni} : H_{uni} \to H$ which satisfies the following property: For all covers $f : H' \to H$ there exists a cover $g : H_{uni} \to H$ such that $f \circ g = f_{uni}$.

  (d) State that universal cover of the triangle is $\mathbb{Z}$ and that of the graph for the hard-core shift is a path with 4 vertices.

  (e) State the result that if H_{uni} is infinite then $Hom(\mathbb{Z}^d, H)$ is not block-gluing. Say that the arguments are very similar to that of the proper 3-colourings being not block-gluing.

  (f) State that it is undecidable whether H_{uni} is finite. The rough argument can be found outlined in [5, Slides 40 and 41]. State the conjecture that SI is probably undecidable.

(5) State the main result of [1] which is Theorem 4.4 (what are the mixing properties for the space of proper colourings) and, if you can, sketch the proof (or a part of it).

Sorry for the long list. I just put down all that I had in mind. But you can pick and choose. The time division among the items can be something like 10, 5, 10, 10, 10 or something like 10, 5, 10, 15, 5 especially if you decide to describe universal covers a bit more and give up on the proof of [1, Theorem 4.4].

References

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