Lecture 6: Computability of entropy for hom-shifts and loads of questions about tiling shifts

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Graph homomorphisms and hom-shifts

A graph homomorphism is a map between graphs which preserves adjacencies. Given a graph H, a hom-shift associated with H is the space

$$X_H^d = \{\text{graph homomorphisms from } \mathbb{Z}^d \text{ to } H\}.$$

There are nearest neighbour shifts of finite type which are invariant under permutations of coordinates and reflections.

Some examples: Full shifts

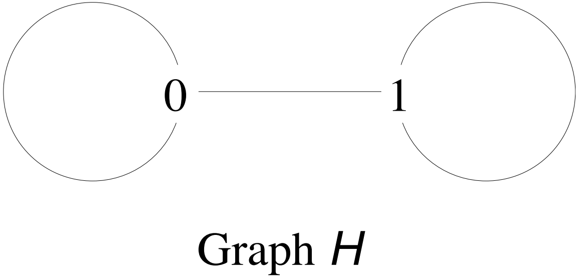
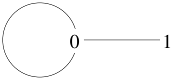


Figure : No Constraints

Some examples: The hard-core shift

The symbols are 0 and 1. Adjacent symbols in this shift cannot be both 1. This is the space X_H^d where H is given by the graph below.



Graph $\mathcal{H}$

1	0	0	0	0
0	0	0	0	0
1	0	1	0	0
0	0	0	1	0
0	1	0	0	0

Some examples: Proper q-colorings

The symbols are $1, 2, 3, \ldots, q$. Adjacent symbols in this shift are distinct. This is the space $X^d_{K_q}$ where K_q is the complete graph on q vertices.

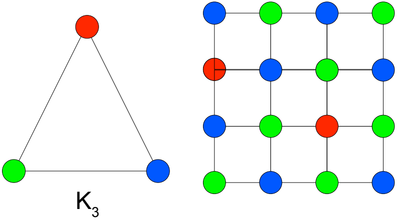


Figure : Proper 3-colorings

Topological entropy

Let X be a shift space and for each n the topological entropy is defined by

$$h_{\mathrm{top}}(X) := \lim_{n\to\infty} \frac{1}{|B_n|} \log(|\mathcal{L}(X, B_n)|).$$

$$h_{\mathrm{top}}(\mathbb{A}^{\mathbb{Z}^d}) = \log(|\mathbb{A}|).$$

Approximation of entropy

Can $h_{\text{top}}(X)$ be computed?

Theorem (Hochman and Meyerovitch (2010))

The set of entropies of SFTs when $d \geq 2$ are precisely the non-negative right recursively enumerable numbers, that is, numbers for which there exists algorithms approximating it from above.

So there is not much hope to do so in complete generality.

Approximation of entropy

BUT

Approximation of entropy

Theorem (Friedland 1997)

The entropy of hom-shifts can be computed, meaning, there exists an algorithm which can give approximating upper and lower bounds.

Entropy of the hard-square shift and an important question

Theorem (Pavlov 2012)

The entropy of the hard-square shift can be approximated upto accuracy $\frac{1}{n}$ *in time* $Poly(n)$.

Question

Is this true for all hom-shifts in 2 *dimensions?*

Entropy of hom-shifts can be computed

Theorem (Friedland 1997)

The entropy of hom-shifts can be computed, meaning, there exists an algorithm which can give approximating upper and lower bounds.

There is a nice and simple argument which shows that for SFTs in general, there is an algorithm which can approximate the entropy from above. So we have to come up with approximating lower bounds.

Why do we love periodic points?

A point $x \in X$ has period $2n$ if $\sigma^{n\vec{e}_i}(x) = x$ for all $1 \leq i \leq d$. The set of configurations with period $2n$ is denoted by

$$Per(X, 2n) := \{x \in X \ : \ \sigma^{2n\vec{e}_i}(x) = x \text{ for all } 1 \leq i \leq d\}.$$

Suppose for a nearest neighbour SFT we know that there a lot of periodic points. More precisely suppose there is a $c > 0$ such that

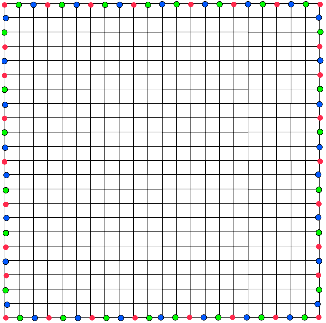
$$\frac{|Per(X, 2n)|}{|\mathcal{L}(X, B_n)|} \geq e^{-cn^{d-1}}.$$

Let us see how we can get an algorithm to approximate the entropy from below.

A nice periodic boundary for each n

Counting periodic points of a nearest neighbour SFT is easy. Let us look at what this means for $d = 2$.

The top row = the bottom row. The left column = the right column and nothing forbidden appears

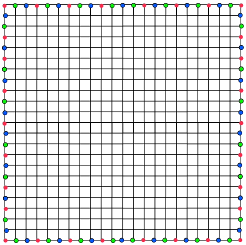


A nice periodic boundary for each n

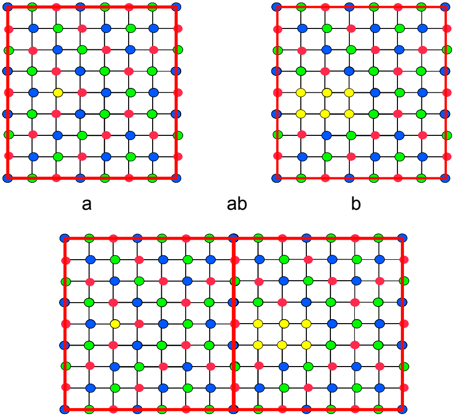
Look at all the elements of $Per(X, 2n)$. There are at most $|\mathbb{A}|^{cn^{d-1}}$ different boundary patterns which can appear.

Thus there exists a boundary pattern a_n such that the number of elements with a_n on the boundary

$$|Per^{a_n}(X, 2n)| \geq |\mathbb{A}|^{-cn^{d-1}} |Per(X, 2n)|$$

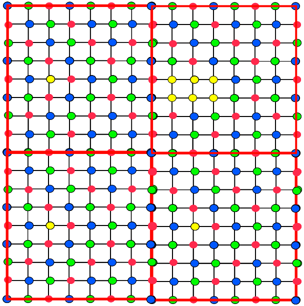


But now notice that if $a, b \in Per^{a_n}(X, 2n)$ then we can put them together side by side and they will still be a valid pattern in X.



Tiling big patterns with smaller ones

B_{kn} can be tiled by k^d disjoint translates of B_n. Each k^d collection of patterns from $Per^{a_n}(X, 2n)$ gives a valid. pattern on B_{kn}.



This gives us an immediate bound (for some $c > 0$)
$|\mathcal{L}(X, B_{kn})| \geq |Per^{a_n}(X, 2n)|^{k^d}$.

Putting it together

We proved:

$$|Per^{a_n}(X,2n)| \geq |\mathbb{A}|^{-cn^{d-1}}|Per(X,2n)|$$
$$|\mathcal{L}(X,B_{kn})| \geq |Per^{a_n}(X,2n)|^{k^d}.$$

Thus

$$|\mathcal{L}(X,B_{kn})| \geq |\mathbb{A}|^{-cn^{d-1}k^d}|Per(X,2n)|^{k^d}.$$

and there exists $c' > 0$ such that

$$\lim_{n\to\infty}\frac{1}{|B_{kn}|}\log(|\mathcal{L}(X,B_{kn})|) \geq \frac{c'}{n} + \liminf_{n\to\infty}\frac{1}{|B_n|}\log(|Per(X,2n)|).$$

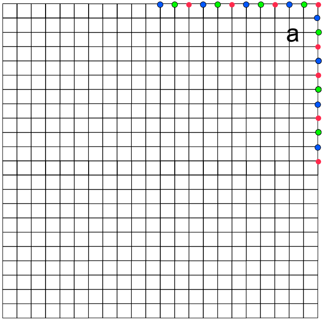
So we will just prove that for hom-shift, X^d_H, there exists $c'' > 0$ such that

$$\frac{|Per(X,2n)|}{|\mathcal{L}(X,B_n)|} \geq e^{-c''n^{d-1}}.$$

How to prove $\frac{|Per(X,2n)|}{|\mathcal{L}(X,B_n)|} \geq e^{-c'' n^{d-1}}$?

Put the uniform probability measure on $\mathcal{L}(X_H^d, B_n)$ and lets restrict our attention to $d = 2$. There must exist some graph homomorphism a from $\partial B_n \cap \mathbb{N}^2$ to H such that

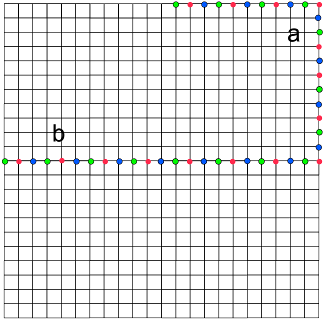
$$\mathbb{P}([a]_{B_n \cap \mathbb{N}^2}) \geq |H|^{-3n}.$$



How to prove $\frac{|Per(X,2n)|}{|\mathcal{L}(X,B_n)|} \geq e^{-c'' n^{d-1}}$? $d = 2$.

It follows that for any graph homomorphism b from $\{-n, -n+1, \ldots, n\} \times \{0\}$ to H,

$\mathbb{P}(a,$ reflection of a about $\{-n, -n+1, \ldots, n\} \times \{0\} \mid b) = \mathbb{P}(a \mid b)^2.$



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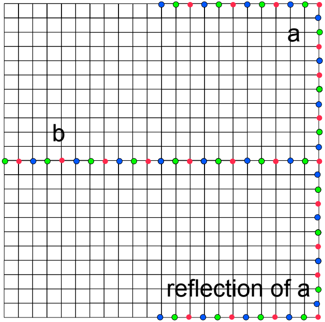
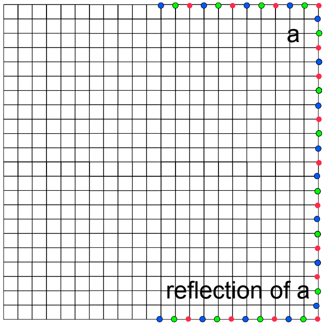


How to prove $\frac{|Per(X,2n)|}{|\mathcal{L}(X,B_n)|} \geq e^{-c'' n^{d-1}}$? $d = 2$

By integrating over all possible values of b, we have that

$$\begin{aligned}\mathbb{P}(a,\ \text{reflection of } a \text{ about } \{-n, -n+1, \ldots, n\} \times \{0\}) &\geq \mathbb{P}(a)^2 \\ &\geq |H|^{-6n}.\end{aligned}$$



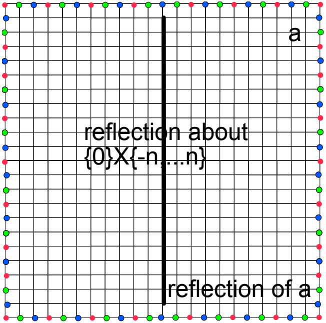
How to prove $\frac{|Per(X,2n)|}{|\mathcal{L}(X,B_n)|} \geq e^{-c'' n^{d-1}}$? $d = 2$

By applying another reflection about $\{0\} \times \{-n, -n+1, \ldots, n\}$ we have that

$$\mathbb{P}(Per(X, 2n)) \geq |H|^{-12n}.$$

Thus

$$\frac{|Per(X, 2n)|}{|\mathcal{L}(X, B_n)|} \geq e^{-c'' n}.$$



A similar idea works in higher dimensions. This is called reflection positivity. A much more advanced application of this method gives us that

1. (Chandgotia and Peled) There exists a fully supported mme for all hom-shifts for connected graphs H. (but we cannot prove that all mmes are fully supported.
2. (Chandgotia and Meyerovitch) Hom-shifts X^d_H are universal when H is not bipartite.

Rectangular tiling shifts

Tilings by rectangular tiles

A rectangular tile is a subset of $\mathbb{Z}^d$ of the form $[1, i_1] \times [1, i_2] \times \cdots \times [1, i_d]$ for $i_1, i_2, \ldots, i_d \in \mathbb{N}$.

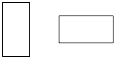
Given a set of rectangular tiles, T, we denote by X_T the set of tilings of $\mathbb{Z}^d$ by elements of T. It comes with the natural $\mathbb{Z}^d$-shift action which makes it a shift of finite type.

Examples: Dimer Tilings

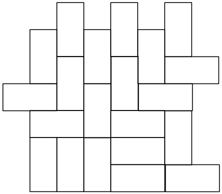
Dimers are the set of rectangular tiles given by

$$T_{dim} = \{[1, i_1] \times [1, i_2] \times \cdots \times [1, i_d] \ : \ \prod_{t=1}^{d} i_t = 2\}.$$

Let X_{dim} be the set of dimer tilings.



Dimers



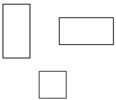
A Dimer tiling

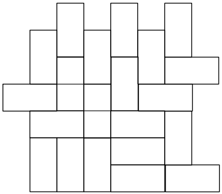
Examples: Monomer k-mer tilings

Let

$$T = \{[1]^d\} \cup \{[1, i_1] \times [1, i_2] \times \cdots \times [1, i_d] \ : \ i_t = 1 \text{ or } k \text{ and } \prod_{t=1}^{d} i_t = k\}$$

The set of tilings by T is called the monomer k-mer shift.

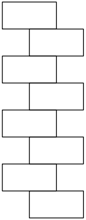




A Monomer 2-mer tiling

Mixing properties of tiling shifts

Dimer tilings are not strongly irreducible.



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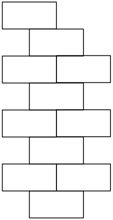

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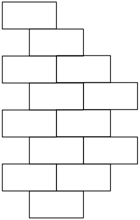

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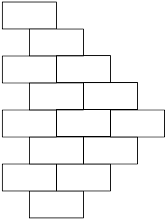
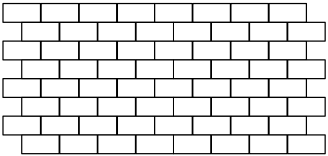


Mixing properties of tiling shifts

However monomer k-mer shifts are clearly strongly irreducible. Given tilings of regions A and B which do not intersect, the complement can be tiled by monomers.



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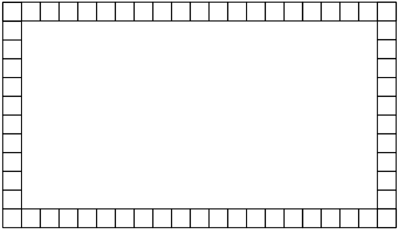

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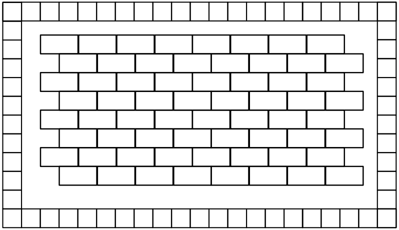

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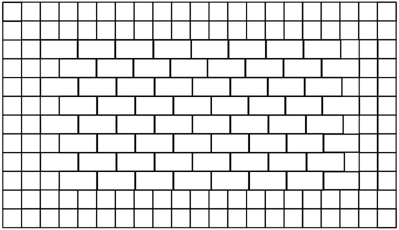


Mixing properties of tiling shifts

Question

Is strong irreducibility of rectangular tiling shifts decidable?

Mixing properties of tiling shifts

A set of tiles T is called prime if the greatest common divisor of side lengths along any given direction is 1.

It is easy to see that if T is not prime then X_T is not topologically mixing.

Theorem (Einsedler 2001)

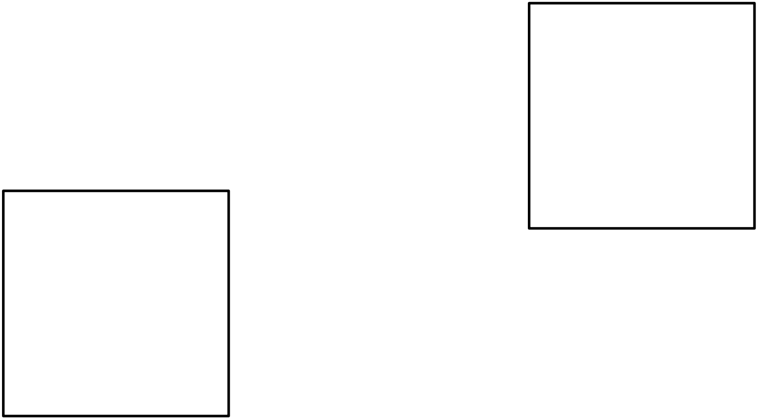
For $\mathbb{Z}^2$ tiling shifts, if $|T| = 2$, then X_T is mixing if and only if T is prime.

Question

Prove this in general.

The main idea for proving mixing

If a set of tiles T is prime, then T can tile the complement of any two rectangles provided the rectangles are far enough apart.



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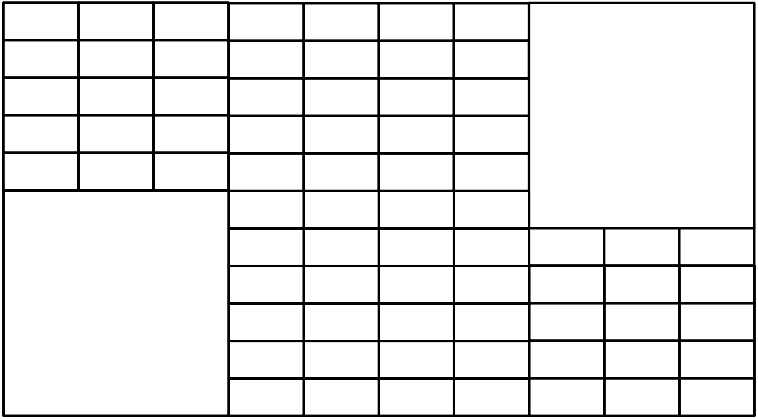
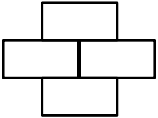


The main idea for proving mixing

So to prove mixing is sufficient to prove that the restriction of any tiling of $\mathbb{Z}^2$ to a finite region can be extended to a tiling of a rectangle.



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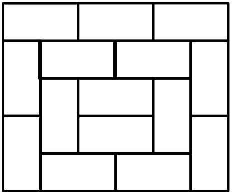


Extending tilings

Question

How often can we extend a tiling of a region to a tiling of a slightly bigger rectangle?

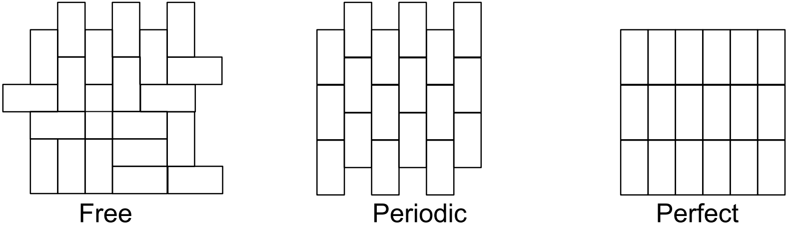
Fix P to be the product of the sides of the tiles in T.

Let $Free_n$ be the set of tilings of $\mathbb{Z}^d$ restricted to $[1, nP]^d$.

Let $Periodic_n$ be the set of nP-periodic tilings of $\mathbb{Z}^d$ restricted to $[1, nP]^d$.

Let $Perfect_n$ be the set of tilings of $\mathbb{Z}^d$ restricted to $[1, nP]^d$.

Topological, Periodic and Perfect



$$h_{top}(X_T) = \lim_{n\to\infty} \frac{1}{(np)^d} \log(Free_n).$$

$$h_{Periodic}(X_T) = \liminf_{n\to\infty} \frac{1}{(np)^d} \log(Periodic_n).$$

$$h_{Perfect}(X_T) = \liminf_{n\to\infty} \frac{1}{(np)^d} \log(Perfect_n).$$

Topological, Periodic and Perfect

We know immediately

$$h_{top}(X_T) \geq h_{Periodic}(X_T) \geq h_{Perfect}(X_T).$$

If T is prime and $h_{top}(X_T) = h_{Perfect}(X_T)$ then X_T is universal.

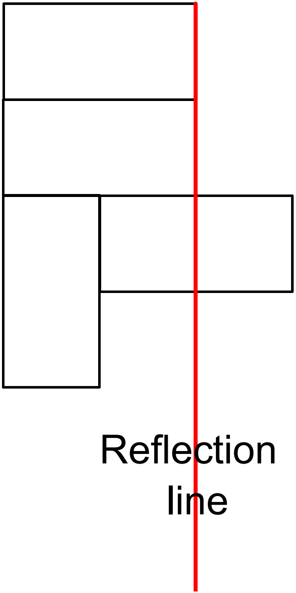
If $h_{top}(X_T) = h_{Periodic}(X_T)$ then $h_{top}(X_T)$ is computable.

For $d = 2$, this follows from Kastelyn's formalism for domino tilings.

Theorem

(C, 2019) For domino tilings $h_{top}(X_T) = h_{Perfect}(X_T)$ for all dimensions d.

The proof uses the fact that dominos can be reflected



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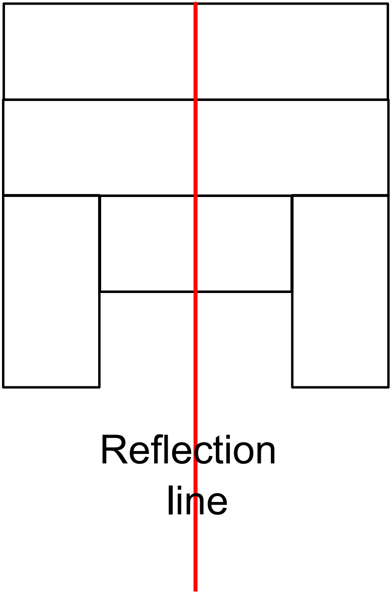

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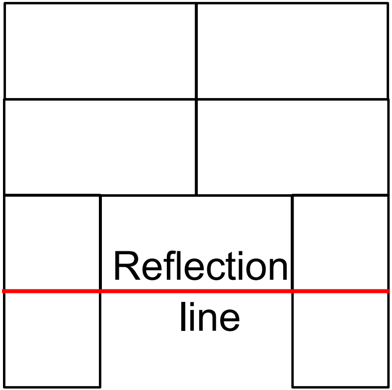

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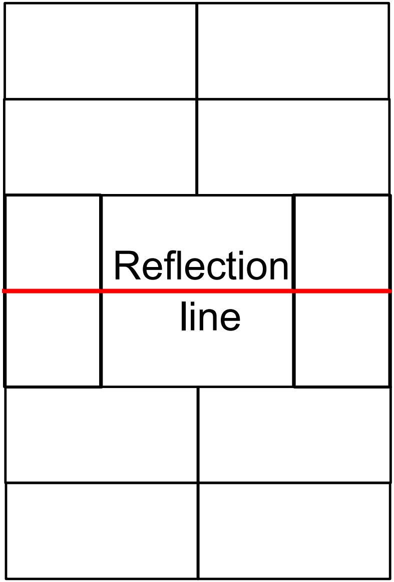

and some $d-1$-cube cohomology.

Conjecture

Conjecture

X_T is universal for all prime tiling sets T. In other words prove that $h_{top}(X_T) = h_{Perfect}(X_T)$.