About Riesz Sets

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Some preliminaries

All measures in this talk will be complex-valued, outer-inner regular, finite and on the group $\mathbb{R}/\mathbb{Z}$ (also denote by $\mathbb{T}$). The Fourier transform of μ is denoted by $\hat{\mu}$.

The Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ will be denoted by μ_l.

The support of a function $f : \mathbb{Z} \to \mathbb{C}$ is given by

$$\operatorname{supp}(f) = \{n \in \mathbb{Z} \ : \ f(n) \neq 0\}.$$

The basic definition

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A set $P \subset \mathbb{Z}$ is called a Riesz set if for all measures μ for which

$$supp(\hat{\mu}) \subset P$$

we have that μ is absolutely continuous with respect to the Lebesgue measure.

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1. If $P = \{p_1, p_2, \ldots, p_n\}$ is a finite set and $\mathrm{supp}(\hat{\mu}) \subset P$, then

$$d\mu(x) = \left(\sum_{t=1}^{n} a_t \exp(2\pi i p_t x) \right) d\mu_l(x)$$

for some $a_t \in \mathbb{C}$. Thus finite sets are Riesz sets.

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1. Finite sets are Riesz sets.
2. Let $P \subset \mathbb{Z}$ be a Riesz set, $\epsilon \in \{-1, 1\}$ and $n \in \mathbb{Z}$. Let us see why $\epsilon P + n$ is also a Riesz set.

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$$d\nu(x) = \exp(2\pi inx)d\mu(\epsilon x).$$

   Then $supp(\hat{\nu}) \subset P$. Since P is a Riesz set, ν is absolutely continuous. Thus μ is absolutely continuous.

Here is a cute exercise. Let $k \in \mathbb{N}$. Prove that P is a Riesz set if and only if kP is a Riesz set.

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Hint: Use the map x to kx from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}/\mathbb{Z}$.

Can infinite sets be Riesz sets?

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This is already a non-trivial question and the answer to this lies in a beautiful theorem by F. and M. Riesz.

Natural numbers form a Riesz set

Theorem (Riesz brothers, 1923)

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An infinite set of natural numbers $\{n_i \ : \ i \in \mathbb{N}\}$ is called a lacunary set if there is a $\lambda > 1$ such that $n_{i+1}/n_i > \lambda$ for all $i \in \mathbb{N}$.

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Theorem (Rudin, 1960)

The union of the negative numbers and a lacunary set is a Riesz set.

In 1968, gave these sets a name and proved many interesting results about them.

Theorem (Meyer 1968)

The union of the negative numbers and the set of squares is a Riesz set.

I do not know if this holds for the cubes. This has much to do with the Bohr topology on the integers which we will discuss soon.

The Main question

The main question I want to advertise via this talk is the following.

We say that a sequence of natural numbers n_i is a sparse sequence if it is an increasing sequence such that the differences $n_{i+1} - n_i$ is also an increasing set

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Here is a partial answer to the question.

Theorem (Wallen, 1970)

Let $P \subset \mathbb{Z}$ be a set of the type given above and μ be a measure such that $supp(\hat{\mu}) \subset P$. Then $\mu \star \mu$ is absolutely continuous.

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Why care?

Why care?

The short answer is that it is a very natural and basic question.

Why care?-Longer answer

A set $Q \subset \mathbb{N}$ is called a totally predictive set for all zero entropy processes $X_i; i \in \mathbb{N}$, X_n is measurable function of $X_i; i \in Q$ for all $n \in \mathbb{Z}$.

In a paper with Benjamin Weiss we proved that

(Under technical assumptions) If Q is a totally predictive set then $\mathbb{Z} \setminus Q$ is a Riesz set.

(Under technical assumptions) If $P \subset \mathbb{N}$ is such that $P \cup (-\mathbb{N})$ is a Riesz set then $\mathbb{N} \setminus P$ is a totally predictive set.

Further, there are many similarities in methods known to prove that a set is predictive and methods known to prove that a set is a Riesz set.

But going too deep into this will take us far afield.

Proof of Riesz brothers' theorem (by Øksendal, 1971)

Something to confuse you with

We will shuttle between the $\mathbb{R}/\mathbb{Z}$ version of the circle and $\{\exp(2\pi ix); x \in [0,1]\}$ model of the circle. For this proof we will use the latter.

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Let μ be a measure such that *supp*$(\hat{\mu}) \subset \mathbb{N}$. We will prove that μ is absolutely continuous.

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Equivalently for all trignometric polynomials p we have that

$$\int p d\mu = 0.$$

This implies for instance that

$$\int \frac{1}{1+rz} d\mu = 0$$

for $r > 1$.

This is what I want you to remember about μ.

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Let $g_n : \mathbb{T} \to \mathbb{C}$ be given by

$$g_n(z) = 1 - \prod_{i=1}^{N} \frac{z - z_i}{z - (1 + nr_i)z_i}.$$

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We will show that g_n approximates the 1_F; the indicator function of F.

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4. Thus $|g_n(z) - 1| \leq \frac{1}{n-1}$ for all $z \in F$.

$g_n(z) = 1 - \prod_{i=1}^{N} \frac{z - z_i}{z - (1 + nr_i) z_i}$: What happens away from F?

Suppose $z \in \mathbb{T}$ such that $|z - z_i| > \delta$ for all $1 \leq i \leq N$ then we have

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&\leq \exp\left(\frac{n \sum_{i=1}^{N} r_i}{\delta} \right) - 1. \\
&\leq \exp\left(\frac{1}{n\delta} \right) - 1 \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Thus $g_n \to 1_F$ as $n \to \infty$. Recall we had $\int g_n d\mu = 0$. By dominated convergence theorem we have $\mu(F) = 0$. This concludes the proof.

Daniel pointed out that the product had a similar form as do Blashke products.

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2. Proof of Riesz brothers' theorem.✓
3. A primer on the Bohr topology.
4. How can Bohr topology help or hurt? Why does this work for the square and not for the cubes. Why can't it work for sparse sequences?
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A primer on the Bohr topology

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It is given the topology generated by the compact-open topology, that is, the topology generated by sets of the form

$$\{\phi : G \to \mathbb{T} \ : \ \phi(K) \subset U\}$$

where $K \subset G$ is compact and $U \subset \mathbb{T}$ is open.

Bohr topology

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Let $\phi_n : \mathbb{T}_d \to \mathbb{T}$ be given by $\phi_n(\alpha) = n\alpha$. Notice the copy of $\mathbb{Z}$ in $\hat{\mathbb{T}}_d$ given by $n \to \phi_n$.

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The Bohr topology is the topology on this copy of $\mathbb{Z}$ induced by $\hat{\mathbb{T}}_d$.

OK! But what does this mean?

But what is Bohr topology?

Given a finite (compact) set $K \subset \mathbb{T}_d$ and an open set $U \subset \mathbb{T}$ let

$$O(K, U) := \{n \in \mathbb{Z} \; : \; \phi_n(K) \subset U\}.$$

But what is Bohr topology?

Given a finite (compact) set $K \subset \mathbb{T}_d$ and an open set $U \subset \mathbb{T}$ let

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Sets of the type $O(K, U)$ generate the Bohr topology. If $K = \alpha_1, \alpha_2, \dots, \alpha_t$ then

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Given $\alpha \in \mathbb{T}^d$ and an open set $U \subset \mathbb{T}^d$ consider the visit times of $x \in \mathbb{T}^d$,

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Clearly this forms the basis for the Bohr topology.

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The main fact that we need about the Bohr topology

Theorem

For all open sets $U \subset \mathbb{Z}$ and $n \in U$, there exists an atomic measure σ such that

$$\hat{\sigma}(n) = 1 \text{ and } \hat{\sigma}|_{\mathbb{Z}\setminus U} = 0.$$

This should remind you of Tietze's extension theorem from topology.

The proof is requires a little bit of background. I have often wondered if there is a simple elementary proof of this fact.

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for all $n \in \hat{\mathbb{T}}_d$.

But if $f \in L^1(\mathbb{T}_d)$ then it is supported on a countable set, that is, $f(x) = \sum_{j\in\mathbb{N}} a_j\delta_{x_j}(x)$.

What is the Fourier transform?

Let G be a locally compact abelian group. By $L^1(G)$ we mean the functions whose modulus is is integrable with respect to a Haar measure μ_{Haar} on the group.

Given $f \in L^1(G)$, we write $\hat{f} : \hat{G} \to \mathbb{C}$ given by

$$\hat{f}(n) = \int_G n(x) f(x) d\mu_{Haar}(x).$$

If $G = \mathbb{T}_d$, the Haar measure is the counting measure and the Fourier transform is just

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for all $n \in \hat{\mathbb{T}}_d$.

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Thus for $n \in \mathbb{Z}$ we have that

$$\hat{f}(n) = \int n(x) f(x) d\mu_{Haar}(x) = \sum_{j \in \mathbb{N}} n(x_j) a_j = \sum_{j \in \mathbb{N}} \exp(2\pi j n x_j) a_j.$$

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We have then that the usual Fourier transform

$$\hat{\mu}_f(n) = \int_{\mathbb{T}} \exp(2\pi inx) d\mu(x) = \sum_{j\in\mathbb{N}} \exp(2\pi jnx_j) a_j = \hat{f}(n).$$

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Now we are ready for the proof.

Atomic measure σ such that $\hat{\sigma}(n) = 1$ and $\hat{\sigma}|_{\mathbb{Z}\setminus U} = 0$ where $n \in U$ is open.

Let $U' \subset \hat{\mathbb{T}}_d$ be an open set such that $U' \cap \mathbb{Z} = U$. Let μ_H denote the Haar (probability) measure on $\hat{\mathbb{T}}_d$.

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By the continuity of the map from $\hat{\mathbb{T}}_d \times \hat{\mathbb{T}}_d \to \hat{\mathbb{T}}_d$ given by $(u, v) \to u - v$ there exists an open set $V \subset \hat{\mathbb{T}}_d$ containing 0 such that $n - (V - V) \subset U'$.

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Equivalently, $(\hat{\mathbb{T}}_d \setminus U') - V$ and $n - V$ are disjoint.

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and for $s \in \mathbb{Z} \setminus U' = \mathbb{Z} \setminus U$ we have that $s - m \notin n - V$ for all $m \in V$ and hence

$$\hat{\sigma}(s) = \frac{1}{\mu(V)} \int_V 1_{n-V}(s-m)\, d\mu(m) = 0.$$

Thus σ is the required atomic measure.

Closures in the Bohr topology

Let us run through some simple examples the closure of Bohr topology.

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But before that we have to introduce some results in ergodic theory.

Poincaré sets

A set $P \subset \mathbb{N}$ is called a Poincaré set if for probability preserving transformations (X, μ, T) and sets U of positive measures, there exists $n \in P$ such that

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Remark: The set $p(\mathbb{Z}) \cap \mathbb{N}$ intersects every Bohr neighbourhood of 0 if and only if p has a root modulo m for every m.

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However for simple irreducible polynomials this is not difficult to prove directly. One can use this to prove the following result.

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Thus

$$\overline{\{n^2 \ : \ n \in \mathbb{N}\}} = \{0\} \cup \overline{\{n^2 \ : \ n \in \mathbb{N}\}}$$

and

$$\overline{\{n^3 \ : \ n \in \mathbb{N}\}} = \{n^3 \ : \ n \in \mathbb{Z}\}.$$

Cliffhanger

Theorem (Ajtai, Havas and Komlós, 1980)

There exists a sequence of natural numbers $n_k; k \in \mathbb{N}$ *which is sparse, meaning,* n_k *is an increasing sequence and the differences* $n_{k+1} - n_k$ *is also an increasing sequence such that*

$$\overline{\{n_k \ : \ k \in \mathbb{N}\}} = \mathbb{Z}.$$

The proof of this is very nice and we can go over it if people are interested.

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Structure of the talk

1. Why did I start caring?✓
2. Proof of Riesz brothers' theorem.✓
3. A primer on the Bohr topology.✓
4. How can Bohr topology help or hurt? Why does this work for the square and not for the cubes. Why can't it work for sparse sequences?
5. Fermat's last theorem and the cubes.
6. How do you prove something is not a Riesz set?
7. Wallen's convolution theorem.

Why do we care about the Bohr topology?

Recalling what we know about Riesz sets

Definition

A set $P \subset \mathbb{Z}$ is called a Riesz set if for all measures μ for which

$$supp(\hat{\mu}) \subset P$$

we have that μ is absolutely continuous with respect to the Lebesgue measure.

A set $P \subset \mathbb{Z}$ is a Riesz set if and only if its reflections and scalings are Riesz sets. The Riesz brother's theorem says that $\mathbb{N}$ form a Riesz set.

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Question

Are there other operations by which we can build new Riesz sets from old ones?

Silly Question

Is the union of two Riesz sets still a Riesz set?

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Of course not. Both N and $\mathbb{Z} \setminus \mathbb{N}$ are Riesz sets? So to take union we need stronger assumptions.

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Here is the main theorem of the entire series of talks.

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If P is a Riesz set and Q is a strong Riesz set then their union $P \cup Q$ is also a Riesz set.

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We just saw that $\overline{\{n^2 \;:\; n \in \mathbb{N}\}} = \{0\} \cup \{n^2 \;:\; n \in \mathbb{N}\} \subset \mathbb{N} \cup \{0\}$ is a strong Riesz set. Hence

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We just saw that $\overline{\{n^3 \ : \ n \in \mathbb{N}\}} = \{n^3 \ : \ n \in \mathbb{Z}\}$. We do not know if it is a strong Riesz set. Thus we do not know if

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If this wasn't bad enough there are sparse sequences $\{n_k \ : \ k \in \mathbb{N}\}$ such that

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As far as I can see this is the only technique known to prove that sets are Riesz sets.

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Given a measure μ on $\mathbb{R}/\mathbb{Z}$ we denote its singular part by μ_s.

Why do we care about atomic measures (How did Bohr topology get in?) ?

Claim: Given a measure μ and an atomic measure σ,

$$(\mu \star \sigma)_s = \mu_s \star \sigma.$$

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Let $f \in L^1(\mu_l)$ be such that $d\mu = \mu_s + f\ d\mu_l$.

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The key idea is that convolution of an atomic measure with an absolutely continuous measure leaves it absolutely continuous and convolution with a singular measure leaves it singular.

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Let μ be a measure such that $supp(\mu) \subset P \cup \overline{Q}$. We need to prove that its singular component, $\mu_s = 0$.

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Let $\nu := \mu \star \sigma$. Now $supp(\hat{\nu}) \subset P$. Since P is a Riesz set,

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If P is a Riesz set and Q is a strong Riesz set then their union $P \cup Q$ is also a Riesz set.

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Let μ be a measure such that $supp(\mu) \subset P \cup \overline{Q}$. We need to prove that its singular component, $\mu_s = 0$. Let $m \in \mathbb{Z} \setminus \overline{Q}$. There exists an atomic measure σ such that $\hat{\sigma}(m) = 1$ and $\hat{\sigma}(n) = 0$ for all $n \in \overline{Q}$.

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$$supp(\hat{\mu}_s) \subset \overline{Q}.$$

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The squares, the cubes and some sparse sequences with large closures

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Let μ be a probability measure and $Q \subset \mathbb{N}$ be such that

1. $\mathbb{N} \setminus Q$ *is* Δ_3^*.
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We do not have any such result for $\Delta_4^\star$ sets (which would cover sparse sequences as well).

Also John suggested that we make a finitary version of Riesz set and see what it means for measures on $\mathbb{Z}/n\mathbb{Z}$. This I am yet to do.

However here is something I promised last time.

There exists a sequence of natural numbers $n_k ; k \in \mathbb{N}$ which is sparse, meaning, n_k is an increasing sequence and the differences $n_{k+1} - n_k$ is also an increasing sequence such that

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Let us see why this is true.

Sparse sets with large closures - Ajtai, Havas and Komlós, 1983

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Note that these n_k's and m_k's can be made sparse.

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The condition on the growth rate of m_k is necessary by results by Pollington (1979) and de Mathan (1980): Lacunary sets are closed in the Bohr topology.

A concentration inequality

Claim: Let $X_k; 1 \leq k \leq K$ be independent complex-valued random variables such that $|X_k| \leq 1$. Let $S_K := \sum_{k=1}^K X_k$. If there exists $\epsilon \in (0,1)$ such that

$$|\mathbb{E}(S_K)| \leq \frac{\epsilon}{4} K$$

then

$$Prob(|S_K| > \epsilon K) < 4 \exp\left(-\frac{\epsilon^2}{100} K\right).$$

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Thus

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So far we had this for real valued random variables. Extension to complex valued random variables is a simple exercise (write $X_k = Y_k + iZ_k$) and use the result for the components.

Key take away from the claim: Control on expectation gives control on probabilities.

An equidistribution result

Let $[n_k, m_k]$ be a sequence of disjoint intervals such that

$$\lim_{k\to\infty} \frac{\log m_k}{k} = 0$$

and $m_k - n_k \to \infty$ as $k \to \infty$. Let a_k be chosen uniformly from the interval $[n_k, m_k]$. Then with probability 1 the following holds: For all $z \in \mathbb{T} \setminus \{1\}$,

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$$|\mathbb{E}(X_k(z))| = \frac{1}{m_k - n_k} \left| \frac{1 - z^{m_k - n_k + 1}}{1 - z} \right| \leq \frac{2}{(m_k - n_k)|1 - z|} \leq \frac{4}{(m_k - n_k)|1 - z|}.$$

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Fix $\delta > 0$. Let

$$q_K := \frac{16}{\delta K} \sum_{k=1}^{K} \frac{1}{m_k - n_k}.$$

Clearly $q_K \to 0$ as $K \to \infty$.

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$$|\mathbb{E}(S_K(z))| \leq \frac{4}{|1-z|} \sum_{k=1}^{K} \frac{1}{m_k - n_k}.$$

Fix $\delta > 0$. Let

$$q_K := \frac{16}{\delta K} \sum_{k=1}^{K} \frac{1}{m_k - n_k}.$$

Clearly $q_K \to 0$ as $K \to \infty$. If $q_K < |1 - z|$ then

$$|\mathbb{E}\left(S_K(z)\right)| \leq \frac{\delta}{4} K$$

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and by concentration inequality,

$$Prob(|S_K(z)| > \delta K) < 4\exp(-\frac{\delta^2}{100} K).$$

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Under the assumption that $q_K < |1-z|$, there exists $j \in \mathbb{N}$ such that for

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$$\begin{aligned}
|S_K(z)| &\leq \left|S_K(z')\right| + \sum_{k=1}^{K} |z'^{a_k} - z^{a_k}| \\
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Choose K large enough such that $m_K < \delta \exp\left\{\frac{\delta^2}{200}K\right\}$ and this would mean that

$$|S_K(z)| \leq |S_K(z')| + \delta K.$$

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$$A_K(\delta) := \{(a_n)_{n\in\mathbb{N}} \ : \ |S_K(z)| > 2\delta K \text{ for some } z; q_K < |1 - z|\}.$$

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For large enough K,

$$Prob(A_K(\delta)) \le 4\exp\left(-\frac{\delta^2}{100}K\right)\exp\left(\frac{\delta^2}{200}K\right)$$

which is summable in K.

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By Borel-Cantelli lemma we have that almost every $(a_n)_{n\in\mathbb{N}}$, there exists $K_0((a_n)_{n\in\mathbb{N}})$ such that for all $K > K_0((a_n)_{n\in\mathbb{N}})$,

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Since q_K is a decreasing sequence we have that for almost every $(a_n)_{n\in\mathbb{N}}$, we have that

$$\limsup_{K\to\infty} \frac{1}{K} \Big| \sum_{k=1}^{K} z^{a_k} \Big| = 0$$

for all $z \in \mathbb{T} \setminus \{1\}$.

Sparse sets with large closures - Ajtai, Havas and Komlós, 1983

Theorem

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Thus there exists a sparse sequence a_k such that for all $z \in \mathbb{T} \setminus \{1\}$,

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What does this have to do with Bohr sets?

From equidistribution to ergodic theorems

Theorem

Let $(a_n)_{n\in\mathbb{N}}$ *be a sequence of natural numbers such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} z^{a_n} = 0$$

for all $z \in \mathbb{T} \setminus \{1\}$*. Let* (X, μ, T) *be a probability preserving transformation,* $\mathcal{I} \subset L^2(\mu)$ *be the set of* L^2 *invariant functions and* $f \in L^2(\mu)$*. Then*

$$\frac{1}{N} \sum_{m=1}^{N} T^{a_m} f \to \mathbb{E}(f \mid \mathcal{I}) \text{ in } L^2(\mu).$$

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Proof.

Let ν_f denote the spectral measure corresponding to f.

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Proof.

Let ν_f denote the spectral measure corresponding to f. Let $\delta : \mathbb{T} \to \mathbb{R}$ denote the function

$$\delta(z) = \begin{cases} 1 \text{ when } z = 1 \\ 0 \text{ otherwise.} \end{cases}$$

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Then we have that

$$||\frac{1}{N}\sum_{m=1}^{N} T^{a_m} f-\mathbb{E}(f\mid\mathcal{I})||_{L^2(\mu)}=||\frac{1}{N}\sum_{m=1}^{N} z^{a_m}-\delta||_{L^2(\nu)}.$$

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Then we have that

$$\left|\left| \frac{1}{N} \sum_{m=1}^{N} T^{a_m} f - \mathbb{E}(f \mid \mathcal{I}) \right|\right|_{L^2(\mu)} = \left|\left| \frac{1}{N} \sum_{m=1}^{N} z^{a_m} - \delta \right|\right|_{L^2(\nu)}.$$

By the dominated convergence theorem, the right hand side converges to 0 and hence so does the left hand side. This proves the required result.

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Choose a Bohr open set U containing 0. Then there is a toral rotation $(\mathbb{T}^d, \alpha)$ and an open set $V \subset \mathbb{T}^d$ such that

$$\{n \in \mathbb{N} \ : \ (n\alpha + V) \cap V \neq \emptyset\} \subset U.$$

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Since n_k satisfies the ergodic theorem we have that

$$(n_k\alpha + V) \cap V \neq \emptyset$$

for some k. Thus $n_k \in U$. This completes the proof.

Structure of the talk

1. Why did I start caring?✓
2. Proof of Riesz brothers' theorem.✓
3. A primer on the Bohr topology.✓
4. How can Bohr topology help or hurt? Why does this work for the square and not for the cubes. Why can't it work for sparse sequences?✓
5. Fermat's last theorem and the cubes.✓
6. How do you prove something is not a Riesz set?
7. Wallen's convolution theorem.

Thank you!